Questions on Last Lecture

Notes:

Why Do we Care about A is similar to B?

1. $C_A(x) = C_B(x)$
2. $m_A(x) = m_B(x)$
3. eigenvalues of A = eigenvalues of B
4. If α is an eigenvalue of A (and hence it is an eigenvalue of B), $\dim(E_\alpha)$ [considering A] $= \dim(E_\alpha)$ [considering B]

$$A = J_3^{(4)} \oplus J_3^{(2)} \oplus J_1^{(5)} \oplus J_1^{(3)}$$

$$m_A(x) = (x-3)^4 (x-1)^5$$

$$C_A(x) = (x-3)^6 (x-1)^8$$

① (Questions and answers on Last Lecture)

Correction: What I called Canonical form, it is known as Companion matrix, so we will stick with this name.

⇒ Q. $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Convince me that A is diagnolizable.

A. By staring, A is the companion matrix of $f(x) = x^3 - x = m_A(x) = C_A(x)$. Since $m_A(x) = x^3 - x = x(x-1)(x+1)$, we know that A is diagnolizable.

⇒ Q. $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$. Convince me that A is not diagnolizable.

A: By staring, A is the companion matrix of $f(x) = x^3 - 2x^2 + x = C_A(x) = m_A(x)$. Since $m_A(x) = x(x-1)^2$, we know by class-Theorem, A is not diagnolizable.

(Doing L.A by staring) (2)

⇒ **Q.** Assume A, 3×3, is symmetric. Will it be possible that $C_A(x) = x(x^2+1)$?

A. No, By class notes, all eigenvalues of A are real.

~~Q. Is A =~~

⇒ **Q.** Is $A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ triangulizable?

A. ~~No~~ yes, by staring, $C_A(x) = m_A(x) = (x^2-1)^2$. Since $m_A(x) = (x-1)^2(x+1)^2$, we conclude that A is triangulizable (class notes).

⇒ **Q.** Is $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ diagnolizable?

Find $\dim(E_2)$ (note $\dim(E_2) = IN(E_2)$)

A. By staring, $A = J_2^{(3)}$. Hence without calculation $\dim(E_2) = 1$.

③ Doing LA by Staring

⇒ Q. Given $C_A(x) = (x-2)^3(x-5)$ and $\dim(E_2) = 2$ (i.e., $IN(E_2) = 2$).

Find $m_A(x)$ and Jordan form of A

A. Let us think and stare at $C_A(x)$.

Since $\dim(E_2) = 2$ and $\dim(E_5) = 1$, we conclude that A is not diagnolizeable. Hence $m_A(x) \neq (x-2)(x-5)$. Thus $m_A(x) = (x-2)^2(x-5)$ or $m_A(x) = (x-2)^3(x-5)$.

Suppose $m_A(x) = (x-2)^2(x-5)$. The Jordan-form of A is $\cancel{J_2^{(2)} \oplus J_5^{(1)}}$ $J_2^{(2)} \oplus J_2^{(1)} \oplus J_5^{(1)}$.

Suppose $m_A(x) = (x-2)^3(x-5)$. Then Jordan-form $J_2^{(3)} \oplus J_5^{(1)}$, this is impossible, since $\dim(E_2) = 2$

Thus $m_A(x) = (x-2)^2(x-5)$ and the Jordan-form of A is $J_2^{(2)} \oplus J_2^{(1)} \oplus J_5^{(1)} \rightarrow$

$$\begin{array}{cl} \left[\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{array}\right] & \begin{array}{l} \rightarrow J_2^{(2)} \\ \rightarrow J_2^{(1)} \\ \rightarrow J_5^{(1)} \end{array} \end{array}$$

④ $\boxed{\text{Doing L.A. by Staring}}$

⇒ **Q.** Given $C_A(x) = (x-1)^3(x-2)$, and $\dim(E_1) = 1$. Find ~~…~~ $m_A(x)$ and the Jordan form of A.

A. Let us ~~state~~ stare at all possible Jordan-Block. Since each Jordan-block contribute only $\underline{1}$ to the dimension of an eigenspace. Clearly $J_1^{(3)} \oplus J_2^{(1)}$ is the Jordan-form of A (note $\dim(E_1) = 1$ and $\dim(E_2) = 1$).

Hence $m_A(x) = C_A(x) = (x-1)^3(x-2)$

So A is similar to

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$\to J_1^{(3)}$.

$\to J_2^{(1)}$

5 $\boxed{\text{Doing Linear Algebra by Staring}}$

Q. Assume $J_3^{(2)} \oplus J_3^{(2)} \oplus J_3^{(1)} \oplus J_2^{(3)}$

is the Jordan ~~+~~ form of a matrix A

Q-A 1) What is the size of A? (Smile and say, clearly ~~8~~ 8x8 by staring at (2)+(2)+(1)+(3)

So A is 8x8.

Q-A 2) Find $C_A(x)$: Clearly $C_A(x) = (x-3)^5 (x-2)^3$.

~~What is~~

3 Find $m_A(x)$.

By staring $m_A(x) = (x-3)^2 (x-2)^3$.

4 Find $\dim(E_3)$ and $\dim(E_2)$.

Answer: $\dim(E_3) = 3$ (note each Jordan block contributes one ~~to~~ the dimension)

$\dim(E_2) = 1$

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Read $\boxed{\text{Doing Linear Algebra by Staring}}$

Q. A, $n \times n$, is nilpotent if $A^k = 0$-matrix for some positive integer k.

Clearly if A is nilpotent, then ~~0 eigenvalues of A~~ 0 is the only eigenvalue of A (for if α is an eigenvalue of A, then α^k is an eigenvalue of A^k, but $A^k = 0$-matrix, so 0 is the only eigenvalue of A).

Find $C_A(x)$.

A. $C_A(x) = x^n$.

~~So we learn that~~

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Q7 Let A be an $n \times n$ matrix and nilpotent. Convince me that $A^n =$ 0-matrix.

A. $C_A(x) = x^n$. By <u>Caley-Hamilton Th</u> we know $C_A(A) = A^n = 0$-matrix.

Q8. Let A, $n \times n$, be idempotent and non zero-matrix. Convince me that $m_A(x) = x-1$ or $m_A(x) = x(x-1)$

A. A is idempotent $\Rightarrow A^2 = A \Rightarrow$ $A^2 - A = 0$-matrix. So let $f(x) = x^2 - x$. Then $f(A) = A^2 - A = 0$-matrix. Hence $m_A(x) = x$ or $m_A(x) = x - 1$ or $m_A(x) = x^2 - x$. Since A is non-zero, $m_A(x) \neq x$. If $A = I_n$, then $m_A(x) = x-1$. If ~~$m_A(x)$~~ $A \neq I_n$, then $m_A(x) = x(x-1)$

Q9. Let A be idempotent, $n \times n$, s.t. $A \neq I_n$ and $A \neq 0$-matrix. ~~Show $C_A(x) = x^2 - x$~~ = ~~$m_A(x)$~~ $\Longrightarrow$ Show $m_A(x) = x^2 - x = x(x-1)$.

Doing L.A. by staring (8)

A- By Question 8, $m_A(x) \neq x$, $m_A(x) \neq (x-1)$. Hence $m_A(x) = x(x-1)$.

Q. Convince me that every non-zero idempotent matrix is diagnolizable.

A: If $A = I_n$, then A is diagonal.
If $A \neq I_n$, then $m_A(x) = x^2 - x = x(x-1)$.
Hence by class Result, A is diagnolizable.

Q. Let A be nonzero idempotent matrix s.t. $A \neq I_n$. Convince me that $C_A(x) = x^k (x-1)^\ell$ s.t. $\ell + k = n$

A- Since $m_A(x) = x^2 - x$ and $m_A(x)$ and $C_A(x)$ have the same roots and $\deg(C_A(x)) = n$, we conclude that $C_A(x) = x^\ell (x-1)^k$ s.t. $k + \ell = n$.

Q. Assume A (5x5) is a nonzero idempotent matrix and $A \neq I_{\cancel{n}\,5}$. Given $\dim(E_0) = 3$. Find the Jordan form of A.

A. We know $m_A(x) = x(x-1)$. Hence A is diagnolizable. Since $\dim(E_0) = 3$, $\dim(E_1) = 2$. ~~Then~~

Thus Jordan-form is

$$J_0^{(1)} \oplus J_0^{(1)} \oplus J_0^{(1)} \oplus J_1^{(1)} \oplus J_1^{(1)}$$

$$\Longrightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow$$

A is similar to this Jordan-form.

~~Q. Stame~~

(10) Doing L.A. by Staring

Q. Stare at this matrix in Jordan form

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

Find $C_A(x)$, $m_A(x)$, $\dim(E_2)$, $\dim(E_5)$

A. By Staring, A is similar to

$$V_2^{(4)} \oplus V_5^{(2)}$$

Hence $C_A(x) = (x-2)^4(x-5)^2$.

$m_A(x) = (x-2)^4(x-5)^2$

$\dim(E_2) = 1$, $\dim(E_5) = 1$

(note each Jordan Block contributes $\underline{1}$ to the dimension).